Binomial Expansion Examples

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$$

Example 1

(a) The expression $(1 - 2x)^4$ can be written in the form $1 + px + qx^2 - 32x^3 + 16x^4$
By using the binomial expansion, or otherwise, find the values of the integers p and q.

(3)

(b) Find the coefficient of x in the expansion of $(2 + x)^9$.

(2)

(c) Find the coefficient of x in the expansion of $(1 - 2x)^4(2 + x)^9$.

(3)

(Total 8 marks)

Example 2

1. (a) Obtain the expansion of $(1 - 2x)^3$

(2)

(b) Hence write down the expansion of

$$\left(1-\frac{2}{x}\right)^3$$

(1)

(c) Find the constant term in the expansion of

$$(1-2x)^3\left(1-\frac{2}{x}\right)^3$$

(3)

(Total 6 marks)

Binomial Expansion Exercise

1. Find the coefficient of the term in x^4 in the binomial expansion of

$$(3 + 2x)^7.$$

(Total 3 marks)

2. Find the coefficient of x^3 in the binomial expansion of $(2 + 3x)^9$. Give your answer as an integer.

(Total 3 marks)

3. (a) Write down all the terms in the binomial expansion of $(1 - x)^5$.

(2)

(b) Find the coefficient of x^3 in the binomial expansion of $(3 - 2x)^5$. Give your answer as an integer.

(2)
(Total 4 marks)

44. Write down the expansion $(1 + x)^7$ in ascending powers of x up to and including the term in x^3.

(3)

Hence determine the value of 1.00001^7 correct to 15 decimal places.

(2)
(Total 5 marks)

5. Find the binomial expansion of $\left(1+\frac{1}{2}x\right)^{16}$ in ascending powers of x up to and including the term in x^3.

(3)

Hence determine the coefficient of x^3 in the expansion of $(23+13x)\left(1+\frac{1}{2}x\right)^{16}$.

(2)
(Total 5 marks)

6. (a) Write down the first four terms in ascending powers of x in the expansion of

$(1 + x)^8$,

simplifying your coefficients as much as possible.

(2)

(b) Find the coefficient of x^3 in the expansion of $(3 - 2x)(1 + x)^8$.

(2)
(Total 4 marks)

Binomial Expansion Solutions

1.	$\binom{7}{4}3^3 2^4$	M1 A1		
	15 120	A1	3	
				[3]
2.	$\binom{9}{3}2^6\ 3^3$	M1 M1		
	145152	A1	3	
				[3]
3.	(a) $1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$	M1		
		A1	2	
	(b) $-10 \times 3^2 \times 2^3 = -720$	M1		
	$3^5\left(\frac{2}{3}\right)^2 \times 10$ *or* $\frac{5\times4\times3}{2\times3}\times3^2\times2^3$ *are acceptable*	A1	2	
	CAO			
	SC *$1080x^2$ 1/2*			
				[4]
4.	$(1 + x)7 \equiv \underline{1 + 7x}$	B1		
	$= 21x^2 + 35x^3$	B1 B1	3	
	Substituting $x = 10^{-5}$	M1		
	$1.00001^7 = \underline{\underline{1.000070002100035}}$ *(to 15 decimal places)*	A1	2	
				[5]

5.	$1 + 8x + 30x^2 +$ (attempt at least 3 terms)	M1	
	may be unsimplified	A1	
	$+ 70x^3$ (simplified)	A1 cao	3
	23 (coeff. of x^3) + 13 (coeff. of x^2)	M1	
	$23 \times 70 + 13 \times 30 = \underline{2000}$	A1 cao	2

[5]

6.	(a) $1 + 8x + 28x^2 + 56x^3$	M1 A1	2
	(b) $28 \times -2 + 3 \times 56$ "their 28 and 56"	M1	
	$= 112$	A1	2

[4]

Binomial Expansion

In the formula booklet, there is a formula for expanding $(1+x)^n$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$$

We can get the values for $\binom{n}{0} \ldots \binom{n}{n}$ using either our calculators or Pascal's triangle

Example

$(1+x)^5$

Once we have this expansion, we can replace x to get other expansions

(a) $(1-x)^5$

(b) $(1+3x)^5$

(c) $\left(1-\frac{x}{2}\right)^5$

We can also do these expansions without having done $(1+x)^n$ first

Example

$(1+2x)^7$

Binomial Expansion

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$$

Expand $(1+x)^6$ using the formula above, and write your answer here:

$(1+x)^6 =$

Now use your answer above to expand the following

	Question	Answer
1	$(1-x)^6$	
2	$(1+2x)^6$	
3	$\left(1+\frac{x}{2}\right)^6$	
4	$(1-3x)^6$	
5		

	$(1-10x)^6$	
6	$(1+x^2)^6$	
7	$\left(1-\frac{x}{10}\right)^6$	
8	$(1-x^3)^6$	

Binomial Expansions

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$$

Question 1

$(1-3x)^4$

Question 2

$\left(1+2x\right)^{8}$

Question 3

$\left(1+\frac{x}{2}\right)^{7}$

Question 4

$\left(1-5x\right)^{3}$

Question 5

$\left(1-\frac{x}{4}\right)^{4}$

Question 6

$\left(1+x^{2}\right)^{7}$

Question 7

$\left(1-x^{3}\right)^{4}$

Binomial Expansion

If we want to expand a bracket that does not begin with a 1, then we have to use a second (slightly harder) formula, that is also in the formula booklet

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Example

$(x+y)^6$

We can now use this expansion to expand other things, by substituting for x and y

(a) $(x+3)^6$

(b) $(2x-y)^6$

We can also use the formula directly to expand a bracket without having done $(x+y)^n$ first

Example

$(2-x)^5$

Binomial Expansion

$$\left(x+y\right)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Use the formula above to expand $\left(x+y\right)^7$ and write your answer here:

$\left(x+y\right)^7$

	Question	Answer
1	$\left(x+2\right)^7$	
2	$\left(x-1\right)^7$	
3	$\left(3x-2\right)^7$	
4	$\left(2x+\frac{1}{2}\right)^7$	

5	$(2-x)^7$	
6	$\left(3-\frac{x}{3}\right)^7$	
7	$(x^2-1)^7$	
8	$\left(\frac{1}{2}-x\right)^7$	

Binomial Expansion

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Question 1

$(2+3x)^4$

Question 2

$$(4x-1)^4$$

Question 3

$$\left(2-\frac{x}{2}\right)^5$$

Question 4

$$\left(5+x^2\right)^3$$

Question 5

$$(2+6x)^3$$

Question 6

$$(x-y)^8$$

Question 7

$$\left(\frac{1}{3}+3x\right)^5$$

Question 8

$\left(x^3-1\right)^5$

Examples of Using Binomial Formulae

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$$

Example 1

Use the appropriate formula to expand $(1-3x)^4$

Example 2

Use the appropriate formula to expand $(2x-y)^3$

Example 3

What is the coefficient of x^7, in the expansion of $(1+2x)^{10}$?

Factorial Notation

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

$${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example 1

Use your calculator to work out the following:

(a) $6!$

(b) ${}^8C_5$

(c) $\binom{10}{8}$

Example 2

Without using a calculator simplify the following:

(a) $\frac{9!}{7!}$

(b) ${}^{n}C_{n-1}$

(c) $\binom{n}{n-2}$

Factorial and Combination Notation (Extension)

WITHOUT USING A CALCULATOR evaluate or simplify the following expressions:

1. ${}^{6}C_{2}$

2. $\dfrac{10!}{8!}$

3. $\dfrac{13!}{12!}$

4. ${}^{8}C_{5}$

5. ${}^{n}C_{n-1}$

6. $\dfrac{n!}{(n-2)!}$

7. $n! \times \dfrac{3!}{(n-1)!} \times \dfrac{7!}{(n-2)!} \times \dfrac{n!}{6!}$

8. $\dfrac{{}^{6}C_{4}}{{}^{3}C_{1}}$

9. ${}^{n-2}C_{n-4}$

10. ${}^{n+1}C_{n-1} + {}^{n-1}C_{n-2} - {}^{n}C_{n-2}$

Factorial and Combination Notation (Extension) Solutions

WITHOUT USING A CALCULATOR evaluate or simplify the following expressions:

1. $^{6}C_{2} = \frac{6\times5}{2} = 15$

2. $\frac{10!}{8!} = \frac{10\times9}{2} = 45$

3. $\frac{13!}{12!} = \frac{13}{1} = 13$

4. $^{8}C_{5} = \frac{8\times7\times6}{3\times2} = 56$

5. $^{n}C_{n-1} = n$

6. $\frac{n!}{(n-2)!} = \frac{n(n-1)}{2}$

7. $n!\times\frac{3!}{(n-1)!}\times\frac{7!}{(n-2)!}\times\frac{n!}{6!}$

$$= \frac{n!}{(n-1)!}\times\frac{n!}{(n-2)!}\times\frac{7!}{6!}\times3!$$

$$= n\times\frac{n(n-1)}{2}\times7\times6 = 21n^{2}(n-1)$$

8. $\dfrac{{}^{6}C_{4}}{{}^{3}C_{1}} = \dfrac{15}{3} = 5$

9. ${}^{n-2}C_{n-4} = \dfrac{(n-2)(n-3)}{2}$

10. ${}^{n+1}C_{n-1} + {}^{n-1}C_{n-2} - {}^{n}C_{n-2}$

$$= \frac{(n+1)n}{2} + (n-1) - \frac{n(n-1)}{2}$$

$$= \frac{n^2}{2} + \frac{n}{2} + n - 1 - \frac{n^2}{2} + \frac{n}{2} = 2n - 1$$

Pascal's Triangle

$n=0$	1
$n=1$	1 1
$n=2$	
$n=3$	
$n=4$	
$n=5$	
$n=6$	
$n=7$	
$n=8$	
$n=9$	
$n=10$	

Pascal's Triangle Questions

Question	Answer
Use Pascal's Triangle to expand the following: $(x+y)^6$	

Now use your answer above to expand the following, simplifying your solution	Answer
$(x+1)^6$	
$(2+y)^6$	
$(1-x)^6$	
$(2x-1)^6$	
$(2a-b)^6$	

Pascal's Triangle Questions (Extension)

The following questions refer to the expansion of $(x+y)^n$.

1. How many terms are there in the expansion of $(x+y)^n$?

2. What is the coefficient of x^n?

3. What is the coefficient of y^n?

4. What is the coefficient of $x^{n-1}y$? And xy^{n-1}?

5. In each term, what is the sum of the power of x and the power of y?

6. Consider the coefficients in the expansion. If we look for the largest coefficient, sometimes there is just one term with this coefficient, and sometimes there are two. What values of *n* give one, and what values give two?

7. For each of the cases in question 6 separately, write (in terms of *n*) the powers of x and y in the term or terms with the largest coefficient.

Pascal's Triangle Questions: Solutions

Question	Answer
Use Pascal's Triangle to expand the following: $(x+y)^6$	$x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$

Now use your answer above to expand the following, simplifying your solution	**Answer**
$(x+1)^6$	$x^6+6x^5+15x^4+20x^3+15x^2+6x+1$
$(2+y)^6$	$64+192y+240y^2+160y^3+60y^4+12y^5+y^6$
$(1-x)^6$	$1-6x+15x^2-20x^3+15x^4-6x^5+x^6$
$(2x-1)^6$	$64x^6-192x^5+240x^4-160x^3+60x^2-12x+1$
$(2a-b)^6$	$64a^6-192a^5b+240a^4b^2-160a^3b^3+60a^2b^4-12ab^5+b^6$

Pascal's Triangle Questions (Extension) Solutions

The following questions refer to the expansion of $(x+y)^n$.

1. How many terms are there in the expansion of $(x+y)^n$?

$n+1$

2. What is the coefficient of x^n?

1

3. What is the coefficient of y^n?

1

4. What is the coefficient of $x^{n-1}y$? And xy^{n-1}?

n (for both terms)

5. In each term, what is the sum of the power of x and the power of y?

n

6. Consider the coefficients in the expansion. If we look for the largest coefficient, sometimes there is just one term with this coefficient, and sometimes there are two. What values of *n* give one, and what values give two?

If *n* is even then there is just one term with the largest coefficient.
If *n* is odd then there are two of these terms

7. For each of the cases in question 6 separately, write (in terms of *n*) the powers of x and y in the term or terms with the largest coefficient.

When *n* is even the term with the largest coefficient is $x^{\frac{n}{2}}y^{\frac{n}{2}}$

When *n* is odd the two terms with the largest coefficient are $x^{\left\lceil \frac{n}{2} \right\rceil}y^{\left\lfloor \frac{n}{2} \right\rfloor}$ and $x^{\left\lfloor \frac{n}{2} \right\rfloor}y^{\left\lceil \frac{n}{2} \right\rceil}$
(The notation $\lfloor x \rfloor$ means the integer just below *x* (or equal to *x* if it is an integer), and the notation $\lceil x \rceil$ means the integer just above *x*)